

High spin limits and non-abelian T-duality

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Synopsis

The action of the non-abelian T-dual of the WZW model is related to an appropriate gauged WZW action via a limiting procedure. We extend this type of equivalence to other σ -models with non-abelian isometries and their non-abelian T-duals, focusing on Principal Chiral models. We reinforce and refine this equivalence by arguing that the non-abelian T-duals are the effective backgrounds describing states of an appropriate parent theory corresponding to divergently large highest weight representations. The proof involves carrying out a subtle limiting procedure in the group representations and relating them to appropriate limits in the corresponding backgrounds. We illustrate the general method by providing several non-trivial examples.

Contents

1	Introduction and conclusions	1
2	Gauged WZW models and non-abelian T-duality	3
3	Solving the wave equation	5
3.1	The limit of infinite highest weight representations	6
4	Example: Non-abelian dual of the $SU(2)$ WZW model	7
4.1	The background geometry	8
4.2	Solving the wave equation	9
4.3	High spin limit and the corresponding effective geometry	11
5	Non-abelian duality in non-isotropic cases	15
5.1	Asymmetric coset reduction	16
6	Example: Non-abelian dual of the $SU(2)$ PCM	18
6.1	Solving the wave equation	19
7	Concluding remarks and future directions	22

1 Introduction and conclusions

An important achievement of string theory is that it can describe spacetime physics at the quantum level beyond the General Theory of Relativity. The most appealing class of models admitting an exact string theoretical description is based on coset G/H conformal field theories (CFTs) [1] that admit a spacetime interpretation via the gauged WZW models [2].

In physical applications one deals with field equations. The generic absence, however, of isometries in the gravitational backgrounds corresponding to G/H coset models makes them unsolvable with any of the traditional methods. This deficiency is not

a problem in low dimensional coset models, such as the prototype example of a two-dimensional black hole in [3], or models in which the subgroup that is being gauged is abelian. It becomes, nevertheless, a major hurdle when the gauge group is non-abelian (see, for instance, [4, 5]).

In recent work we developed a method that overcomes this problem using techniques based on the rich, albeit not manifest, underlying group theoretic structure [6]. We gave the general procedure and, in addition, we presented explicit results for the background corresponding to the $SU(2)_{k_1} \times SU(2)_{k_2} / SU(2)_{k_1+k_2}$ model.

In our present work we focus on the sector of the theory corresponding to representations with divergently large values of highest weight. This is a consistent sector and admits a description in terms of an effective gravitational background, provided that a correlated limit in the levels is taken so that the eigenenergies of the theory remain finite. Based on the specific $SU(2)$ example mentioned above, there are indications [6] that these effective gravitational backgrounds are related to the so-called non-abelian T-duals of the WZW backgrounds. This is further supported by the fact that the gauged WZW action for the coset $(G_k \times H_\ell) / H_{k+\ell}$ is equivalent in the $\ell \rightarrow \infty$ limit, to the action for the non-abelian T-dual of the WZW model for G_k with respect to the subgroup H [7].

In the present paper we reinforce this relation by considering the above limit at the level of the states of the theories. Specifically, we construct the eigenstates of the scalar equation for the background fields of the coset theory and carefully take the large spin limit. We demonstrate that these states solve the scalar wave equation for the effective limiting background, or, equivalently, for the non-abelian dual of the original WZW model for G_k . We also extend this equivalence to other σ -models with non-abelian isometries and their non-abelian T-duals focusing, in particular, on Principal Chiral models. In our discussion we present general arguments and give explicit results.

Our results improve our understanding of non-abelian T-duality [8, 9, 10] which, unlike the abelian T-duality originating in a string context in [11], has remained in comparison rather poorly understood in spite of a substantial body of work, e.g. [12]-[20]. In particular, one may now consider these transformations as generating effective backgrounds for describing consistent sectors of some parent theories in the limit of infinite highest weight representations. In fact, this is the physical reason for the

fact that the non-abelian T-duality transformation is non-invertible at the level of its path integral formulation.

2 Gauged WZW models and non-abelian T-duality

In this section we briefly review the relation of the gauged WZW models and the non-abelian duals of WZW models at the level of their classical actions.

Consider coset models of the type $(G_k \times H_\ell)/H_{k+\ell}$, with the subgroup H appropriately embedded into the direct product of the groups $G \times H$. The gauged WZW action is [2]

$$\begin{aligned}
S_{\text{gWZW}}(g, h, A_\pm) &= kI_0(g) + \ell I_0(h) \\
&+ \frac{1}{\pi} \int_M \text{Tr} \left[kA_- \partial_+ g g^{-1} + \ell A_- \partial_+ h h^{-1} - kA_+ g^{-1} \partial_- g \right. \\
&\quad \left. - \ell A_+ h^{-1} \partial_- h + kA_- g A_+ g^{-1} + \ell A_- h A_+ h^{-1} - (k + \ell) A_- A_+ \right], \tag{2.1}
\end{aligned}$$

where g and h are elements of the groups G and H , respectively, parametrized by a total of $\dim(G) + \dim(H)$ variables X^M , and $I_0(g)$ and $I_0(h)$ are the corresponding WZW actions. The gauge fields A_\pm also take values in the Lie algebra of H and the above action is invariant under the gauge transformations

$$g \rightarrow \Lambda^{-1} g \Lambda, \quad h \rightarrow \Lambda^{-1} h \Lambda, \quad A_\pm \rightarrow \Lambda^{-1} A_\pm \Lambda - \Lambda^{-1} \partial_\pm \Lambda, \tag{2.2}$$

for a group element $\Lambda(\sigma^+, \sigma^-) \in H$.

The procedure of obtaining a σ -model from (2.1) involves two steps. Due to the gauge invariance we may gauge-fix $\dim(H)$ parameters in g and h , thus reducing the number of parameters to $\dim G$, thereafter denoted by X^μ . The gauge fields A_\pm can be integrated out via their equations of motion, yielding a σ -model action with a metric $G_{\mu\nu}$, an antisymmetric tensor $B_{\mu\nu}$ and a dilaton field Φ . One can give general expressions for all these fields, but this will not be needed for our purposes.

Let us now turn to non-abelian T-duality applied on WZW actions for some group G .

To perform such a transformation on a WZW action we start with the action

$$S_{\text{nonab}}(g, v, A_{\pm}) = kI_0(g) + \frac{k}{\pi} \int_M \text{Tr} \left[A_- \partial_+ g g^{-1} - A_+ g^{-1} \partial_- g + A_- g A_+ g^{-1} - A_- A_+ \right] - i \frac{k}{\pi} \int_M \text{Tr}(v F_{+-}) , \quad (2.3)$$

where the field strength for the gauged fields is defined as

$$F_{+-} = \partial_+ A_- - \partial_- A_+ - [A_+, A_-] . \quad (2.4)$$

The first line in (2.3) is the usual gauged WZW action for a group G with respect to the vector action of a subgroup H . The second line is just a Lagrange multiplier, with the corresponding fields v in the Lie algebra of H , which forces the field strength F_{+-} to vanish. The above action is invariant under the gauge transformations

$$g \rightarrow \Lambda^{-1} g \Lambda , \quad v \rightarrow \Lambda^{-1} v \Lambda , \quad A_{\pm} \rightarrow \Lambda^{-1} A_{\pm} \Lambda - \Lambda^{-1} \partial_{\pm} \Lambda , \quad F_{+-} \rightarrow \Lambda^{-1} F_{+-} \Lambda , \quad (2.5)$$

again for a group element $\Lambda(\sigma^+, \sigma^-) \in H$, which is similar to (2.2).

The dual backgrounds are obtained as in the case of the abelian T-duality [11]. If we first integrate out the Lagrange multipliers v , then this forces $F_{+-} = 0$ which means that locally the gauge fields A_{\pm} can be set to zero, resulting to the standard WZW action for the group G . Alternatively, we may integrate over the gauge fields (after we partially integrate the vF -term), as they appear non-dynamically, obtaining a different σ -model action. The gauge invariance (2.5) can be used to gauge fix $\dim(H)$ parameters among the total of $\dim(G) + \dim(H)$ parameters in g and in v . If H is a proper subgroup of G , then one could choose to gauge fix all parameters among those in g . If $H = G$, then necessarily some of the v 's are gauged fixed as well. In any case, the maximum number of v 's that can be fixed is $\dim(H) - \text{rank}(H)$.

It was shown in [7] that in the limit $\ell \rightarrow \infty$ the gauged WZW model action (2.1) reduces to the action for the non-abelian duality (2.3), provided an appropriate limiting procedure is followed. We presently review the essential features for our purposes. Let's rescale the variables parametrizing the group element $h \in H$ with ℓ so that in the

limit of large ℓ we have the infinitesimal expansion around the identity as

$$h = I + i \frac{k}{\ell} v + \mathcal{O}\left(\frac{1}{\ell^2}\right) . \quad (2.6)$$

Substituting into (2.1) we obtain (2.3) with the correct Lagrange multiplier term. We note that the WZW part of the action $I_0(h)$ does not contribute at all in the above limit. Hence, at the level of the classical action we have the relation

$$\left. \frac{G_k \times H_\ell}{H_{k+\ell}} \right|_{\ell \rightarrow \infty} = \text{dual of } G_k \text{ with respect to } H \text{ (vector)} . \quad (2.7)$$

Of course, taking this limit in the background that correspond to the action (2.1) (after a gauge fixing and elimination of the gauge fields) is an equivalent procedure as we will demonstrate in the example of section 4.

One way of thinking of the above limiting procedure is that we focus and explore the area around the identity element of the group. Since this process is classically well defined, we expect that the non-abelian T-dual background effectively describes a consistent sector of states of the original theory. Demonstrating this will put the classical equivalence to a firmer quantum mechanical footing.

3 Solving the wave equation

The σ -models corresponding to general coset models are quite complicated and they lack isometries. In [6] we developed a general systematic method to solve the field equations for the associated background fields, based on the underlying group theoretical structure. We focused for concreteness on the scalar field equation, which in a background with metric $G_{\mu\nu}$ and dilaton Φ is of the form

$$-\frac{1}{e^{-2\Phi}\sqrt{G}} \partial_\mu e^{-2\Phi} \sqrt{G} G^{\mu\nu} \partial_\nu \Psi = E \Psi . \quad (3.1)$$

To obtain the general solution to this equation one starts with the irreducible representations (irreps) of $G \times H$, given by direct products $R \times r$. Then the eigenstates of the Laplacian on the full group manifold are (the matrix indices μ, ν in r below not to

be confused with the spacetime indices μ, ν above)

$$R_{\alpha\beta}(g) r_{\mu\nu}(h) , \quad (3.2)$$

with eigenvalues

$$E(R, r) = \frac{C_2(R)}{k + g_G} + \frac{C_2(r)}{\ell + g_H} . \quad (3.3)$$

Under the vector H -transformation the above states transform in the representation

$$(R \times r) \times (\bar{R} \times \bar{r}) = (r_1 \oplus r_2 \oplus \cdots) \otimes (\bar{r}_1 \oplus \bar{r}_2 \oplus \cdots) , \quad (3.4)$$

where on the right hand side we decomposed $R \times r$ and its conjugate into irreps r_i of H . We get a singlet from all products of the form $r_i \times \bar{r}_i$. Denoting by $C_{\alpha\mu}^a(R, r; r_i)$ the Clebsch–Gordan coefficient projecting the state α of R and the state μ of r into the state a of r_i , we construct coset eigenstates as

$$\psi_{R, r; r_i}(g, h) = \sum_{a; \alpha, \beta, \mu, \nu} C_{\alpha\mu}^a(R, r; r_i) C_{\beta\nu}^a(R, r; r_i) R_{\alpha\beta}(g) r_{\mu\nu}(h) , \quad (3.5)$$

with eigenvalues

$$E(R, r; r_i) = \frac{C_2(R)}{k + g_G} + \frac{C_2(r)}{\ell + g_H} - \frac{C_2(r_i)}{k + \ell + g_H} , \quad (3.6)$$

where g_G and g_H are the dual Coxeter numbers for G and H .

By construction the eigenfunctions do not depend on k and ℓ , so we restrict ourselves to the semiclassical limit in which the dual Coxeter numbers are ignored and the backgrounds simplify considerably.

3.1 The limit of infinite highest weight representations

Consider representations r of the Lie-algebra for H with high values of the highest weight which we will denote by j . Then necessarily the irreps r_i in its tensor product with R in the Lie algebra of G (of finite highest weight) have also values for their highest weight of order j . Then for $j \gg 1$ we may write

$$C_2(r) = a(r)j^2 + b(r)j + \mathcal{O}(1) , \quad (3.7)$$

where the highest power of j is dictated by the fact that the Casimir operator is a quadratic one. There is a similar expression for $C_2(r_i)$ with coefficients $a(r_i) = a(r)$ and $b(r_i)$, but with j replaced by $j + n$, where n is finite. In the infinite j limit, the eigenvalues (4.17) become infinite unless the level ℓ becomes infinite as well in a way proportional to j . Specifically, let in a convenient parametrization

$$\ell = \frac{k}{\delta} j, \quad (3.8)$$

where δ is a real positive number. Then from (3.6) we obtain that

$$E(R, b(r), b(r_i)) = \lim_{j \rightarrow \infty} E(R, r; r_i) = \frac{C_2(R)}{k} + \frac{a(r)(\delta - 2n) + b(r) - b(r_i)}{k} \delta. \quad (3.9)$$

Taking the limit in the eigenfunction (3.5) is more delicate since it involves the limiting behaviour of the Clebsch–Gordan coefficients, as well as of the representations. The latter actually should be such that they blow up the region around part of the manifold in a way that the background fields have a well defined limit as well.

In some sense this limiting procedure is similar to the Penrose limit in which the geometry around a null geodesic is explored, resulting to a plane wave. Indeed, the Penrose limit within the context of the AdS/CFT correspondence also involves a restriction to high spin sectors of the appropriate gauge theories [21]. However, unlike the Penrose limit that necessarily requires Minkowski signature, in the present case we may have Euclidean signature backgrounds as well. In both cases, even if the original background geometry is compact, this property is lost in the limit.

4 Example: Non-abelian dual of the $SU(2)$ WZW model

We will test the above general ideas in the case of the coset $SU(2)_{k_1} \times SU(2)_{k_2} / SU(2)_{k_1+k_2}$, for which the general gauged WZW action (2.1) for direct product groups can be used.

4.1 The background geometry

We first review the construction in [6]. We parametrize the associated group elements in the fundamental representation as

$$g_1 = \begin{pmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{pmatrix}, \quad g_2 = \begin{pmatrix} \beta_0 + i\beta_3 & \beta_2 + i\beta_1 \\ -\beta_2 + i\beta_1 & \beta_0 - i\beta_3 \end{pmatrix}, \quad (4.1)$$

where from unitarity

$$\alpha_0^2 + \vec{\alpha}^2 = 1, \quad \beta_0^2 + \vec{\beta}^2 = 1. \quad (4.2)$$

We also note the following parametrization for a group element $g \in SU(2)$

$$g = e^{\frac{i}{2}(\phi_1 - \phi_2)\sigma_3} e^{\frac{i}{2}\theta\sigma_2} e^{\frac{i}{2}(\phi_1 + \phi_2)\sigma_3} = \begin{pmatrix} \cos \frac{\theta}{2} e^{i\phi_1} & \sin \frac{\theta}{2} e^{-i\phi_2} \\ -\sin \frac{\theta}{2} e^{i\phi_2} & \cos \frac{\theta}{2} e^{-i\phi_1} \end{pmatrix}, \quad (4.3)$$

which is also the fundamental $j = 1/2$ representation and where the Euler angles are $\phi = \phi_1 - \phi_2$ and $\psi = \phi_1 + \phi_2$.

We gauge the diagonal $SU(2)$ subgroup of the full $SU(2) \times SU(2)$ group. Under this, $\vec{\alpha}$ and $\vec{\beta}$ transform as vectors. The background depends only on invariants of these three-vectors. They can be chosen to be the three combinations

$$\alpha = |\vec{\alpha}|, \quad \beta = |\vec{\beta}|, \quad \gamma = \vec{\alpha} \cdot \vec{\beta},$$

$$0 \leq \alpha, \beta \leq 1, \quad |\gamma| \leq \alpha\beta. \quad (4.4)$$

Then, by following the general procedure, we obtain a σ -model with metric

$$ds^2 = \frac{k_1 + k_2}{(1 - \alpha_0^2)(1 - \beta_0^2) - \gamma^2} (\Delta_{\alpha\alpha} d\alpha_0^2 + \Delta_{\beta\beta} d\beta_0^2 + \Delta_{\gamma\gamma} d\gamma^2$$

$$+ 2\Delta_{\alpha\beta} d\alpha_0 d\beta_0 + 2\Delta_{\alpha\gamma} d\alpha_0 d\gamma + 2\Delta_{\beta\gamma} d\beta_0 d\gamma), \quad (4.5)$$

where

$$\Delta_{\alpha\alpha} = \frac{(1+r)^2 - r(2+r)\beta_0^2}{r(1+r)^2}, \quad \Delta_{\beta\beta} = \frac{(1+r^{-1})^2 - r^{-1}(2+r^{-1})\alpha_0^2}{r^{-1}(1+r^{-1})^2},$$

$$\Delta_{\gamma\gamma} = \frac{1}{2+r+r^{-1}}, \quad \Delta_{\alpha\beta} = \gamma + \frac{\alpha_0\beta_0}{2+r+r^{-1}}, \quad (4.6)$$

$$\Delta_{\alpha\gamma} = -\frac{\beta_0}{(1+r)^2}, \quad \Delta_{\beta\gamma} = -\frac{\alpha_0}{(1+r^{-1})^2},$$

with $r = k_2/k_1$. The antisymmetric tensor is zero and the dilaton reads (up to a constant)

$$e^{-2\Phi} = (1 - \alpha_0^2)(1 - \beta_0^2) - \gamma^2. \quad (4.7)$$

The background is manifestly invariant under the interchange of α_0 and β_0 and a simultaneous inversion of the parameter r . This symmetry interchanges the two $SU(2)$ s.

4.2 Solving the wave equation

We present here the general solution of the scalar equation by specializing the general formula (3.5) and discussion to our case. A general representation R^j of $GL(2, \mathbb{R})$ has matrix elements [22]

$$R_{m_1, m_2}^j(a, b, c, d) = \sum_k A_{m_1, m_2, k}^j a^{j-m_1-k} d^{j+m_2-k} b^k c^{k+m_1-m_2}, \quad (4.8)$$

where

$$A_{m_1, m_2, k}^j = \frac{\sqrt{(j+m_1)!(j-m_1)!(j+m_2)!(j-m_2)!}}{k!(j-m_1-k)!(j+m_2-k)!(k+m_1-m_2)!}. \quad (4.9)$$

The summation over k extends to all values for which the factorials have non-negative arguments. For instance, (4.8) reproduces the fundamental representation of $GL(2, \mathbb{R})$

$$R^{1/2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \quad (4.10)$$

For the group $SU(2)$ that we are specifically interested, the fundamental representation $R^{1/2}$ is identified with (4.3). In addition, j is a half-integer and $m_1, m_2 = -j, -j+1, \dots, j$. Then, the above sum is finite and the integer k ranges between the extreme values $\max(0, m_2 - m_1)$ and $\min(j + m_2, j - m_1)$. For the $SU(2)$ case it is customary to use the notation $D_{m_1, m_2}^j(\phi, \theta, \psi)$ for the irreps, which are the so-called D -functions. Using the parametrization (4.3) these can be expressed as

$$D_{m_1, m_2}^j(\phi, \theta, \psi) = e^{-i(m_1\phi + m_2\psi)} d_{m_1, m_2}^j(\theta), \quad (4.11)$$

where the Wigner's d -matrix can be written in terms of the Jacobi polynomials. Introducing for notational convenience the non-negative integers

$$m = |m_1 - m_2|, \quad n = |m_1 + m_2|, \quad (4.12)$$

one can prove that

$$d_{m_1, m_2}^j(\theta) = \sqrt{\frac{(j + m/2 + n/2)! (j - m/2 - n/2)!}{(j - m/2 + n/2)! (j + m/2 - n/2)!}} \times \left(\sin \frac{\theta}{2}\right)^m \left(\cos \frac{\theta}{2}\right)^n P_{j-m/2-n/2}^{m,n}(\cos \theta), \quad (4.13)$$

where we should also insert a minus sign after the equality if $m_1 - m_2$ is an odd positive integer. The normalization of the Wigner functions is such that

$$\int_0^\pi d\theta \sin \theta d_{m_1, m_2}^i(\theta) d_{m_1, m_2}^j(\theta) = \frac{\delta_{i,j}}{j + 1/2}. \quad (4.14)$$

Since we have two $SU(2)$ factors we label the corresponding representations $R^{j_1}(g_1)$ and $R^{j_2}(g_2)$, where g_1 and g_2 are the fundamental representations parametrized as in (4.1). Then the general state is [6]

$$\Psi_{j_1, j_2}^j = \sum_m \sum_{m_2, n_2 = -j_2}^{j_2} C_{j_1, m - m_2, j_2, m_2}^{j, m} C_{j_1, m - n_2, j_2, n_2}^{j, m} R_{m - m_2, m - n_2}^{j_1}(g_1) R_{m_2, n_2}^{j_2}(g_2), \\ -\min(j_1 - m_2, j_1 - n_2, j) \leq m \leq \min(j_1 + m_2, j_1 + n_2, j). \quad (4.15)$$

where the $C_{j_1, m_1, j_2, m_2}^{j, m}$ are the Clebsch–Gordan coefficients for a state $|j, m\rangle$ in the diagonal $SU(2)_L$ composed from states $|j_1, m_1\rangle |j_2, m_2\rangle$ in $SU(2)_L \times SU(2)_L$. Similarly, the $C_{j_1, n_1, j_2, n_2}^{j, m}$ are the Clebsch–Gordan coefficients for a state $|j, m\rangle$ in the diagonal $SU(2)_R$ composed from states $|j_1, n_1\rangle |j_2, n_2\rangle$ in $SU(2)_R \times SU(2)_R$. The sum is formed in such a way that a singlet of the diagonal $SU(2)_L \times SU(2)_R$ is obtained. The explicit expression for the Clebsch–Gordan coefficients is [23]

$$C_{j_1, m_1, j - n_1, m - m_1}^{j, m} = \sum_k (-1)^k \left(\frac{2j + 1}{2j + 1 + j_1 - n_1} \right)^{1/2} \\ \times \frac{[(j_1 - n_1)!(j_1 + n_1)!(j_1 + m_1)!(j_1 - m_1)!]^{1/2}}{k!(j_1 - m_1 - k)!(n_1 + m_1 + k)!(j_1 - n_1 - k)!} \\ \times \left(\frac{(2j - j_1 - n_1)!(j + m - n_1 - m_1)!(j - m - n_1 + m_1)!(j + m)!(j - m)!}{(2j + j_1 - n_1)![(j + m - n_1 - m_1 - k)!(j - m - j_1 + m_1 + k)!]^2} \right)^{1/2}, \quad (4.16)$$

where the summation extends to all values for which the arguments of the factorials are non-negative. Note that we have introduced the half integer n_1 to parametrize the

deviation of the spin j_2 from j , obeying $|n_1| \leq j_1$.

The state (4.15) has an eigenvalue (in the semiclassical regime) equal to

$$E_{j_1, j_2}^j = \frac{j_1(j_1 + 1)}{k_1} + \frac{j_2(j_2 + 1)}{k_2} - \frac{j(j + 1)}{k_1 + k_2}. \quad (4.17)$$

Given a pair of values for (j_1, j_2) there are $2j_{\min} + 1$ values for j , where j_{\min} is the minimum of the j_i 's. The above eigenstates are orthogonal for different values of the triad (j_1, j_2, j) with respect to the measure

$$e^{-2\Phi} \sqrt{G} d\alpha_0 \wedge d\beta_0 \wedge d\gamma \sim d\alpha_0 \wedge d\beta_0 \wedge d\gamma. \quad (4.18)$$

4.3 High spin limit and the corresponding effective geometry

Of particular interest is the large spin behaviour. We assume that one of the spins becomes large, whereas the other one is kept finite. For instance, consider

$$j_1 \gg 1, \quad j_2 = \text{finite} \quad \implies \quad j \gg 1. \quad (4.19)$$

In this limit, the eigenvalues (4.17) become infinite unless the level k_1 becomes large as well, but in a way proportional to j . Specifically, let

$$j_1 = j - n, \quad k_1 = \frac{k_2}{\delta} j, \quad (4.20)$$

where n is a half-integer and δ a positive real parameter. Then

$$E_{j_2, n, \delta} = \lim_{j \rightarrow \infty} E_{j_1, j_2}^j = \frac{j_2(j_2 + 1)}{k_2} + \frac{\delta - 2n}{k_2} \delta, \quad (4.21)$$

in accordance with the general result (3.9). Taking the level $k_1 \rightarrow \infty$ has implications for the geometry supporting these infinite spin states. It is straightforward to show that in order for the background to have a good limiting behaviour one should focus on a neighborhood of the manifold.

We focus around $\alpha_0 = 1$ and $\gamma = 0$, by performing first the coordinate transformation

$$\alpha_0^2 = 1 - r^2 \left[(x_1 + \psi)^2 + x_3^2 \right], \quad \gamma = r(x_1 + \psi) \cos \psi, \quad \beta_0 = \sin \psi, \quad (4.22)$$

followed by the limit $r \rightarrow 0$. Then the new variables x_1 and x_3 that we will use, instead

of α_0 and γ , become uncompactified. In this limit we obtain for the metric and dilaton

$$\begin{aligned} ds^2 &= k_2 \left(d\psi^2 + \frac{\cos^2 \psi}{x_3^2} dx_1^2 + \frac{(x_3 dx_3 + (\sin \psi \cos \psi + x_1 + \psi) dx_1)^2}{x_3^2 \cos^2 \psi} \right), \\ e^{-2\Phi} &= x_3^2 \cos^2 \psi. \end{aligned} \quad (4.23)$$

The above can be considered as the effective background describing the high spin sector of the original CFT coset model. It is also the non-abelian T-dual of the $SU(2)$ WZW model with respect to $SU(2)$ (after some renaming of variables it becomes identical to eqs. (6.10) and (6.11) of [12])

Next we would like to explicitly demonstrate that the general state (4.15) has a well defined large spin limit that simultaneously solves the scalar wave equation corresponding to the above limit background. In this respect we first consider the asymptotic behavior of the Clebsch-Gordan coefficients. Using Stirling's formula one may prove from (4.16) the following limit [24]

$$\lim_{j \rightarrow \infty} C_{j-n, m-m_2, j_2, m_2}^{j, m} = d_{m_2, n}^{j_2}(\zeta), \quad \lim_{j \rightarrow \infty} C_{j-n, m-n_2, j_2, n_2}^{j, m} = d_{n_2, n}^{j_2}(\zeta), \quad \cos \zeta = \frac{m}{j}, \quad (4.24)$$

where $d_{m_2, n}^{j_2}$ and $d_{n_2, n}^{j_2}$ are Wigner's d -matrix given in (4.13). Note that in this limit we have by assumption that

$$j_2, m_2, n_2, n = \text{finite}. \quad (4.25)$$

Hence, remarkably, in the large spin limit the Clebsch-Gordan coefficients do not trivialize but get associated with an auxiliary $SU(2)$ irrep of spin equal to the smallest of the three spins that enter in the Clebsch-Gordan coefficients. In addition, the high spin limit turns the summation over m in (4.15) into an integration over the angular variable $0 \leq \zeta \leq \pi$.

Next we determine the expression for the irrep matrix $R_{m-m_2, m-n_2}^{j-n}(g_1)$ that enters in (4.15). To do so we take advantage of the freedom to fix the gauge appropriately so as to facilitate the evaluation of (4.8). We chose the gauge

$$\alpha_1 = \alpha_2 = \beta_2 = 0, \quad (4.26)$$

so that the remaining entries are

$$\alpha_3 = \alpha, \quad \beta_3 = \frac{\gamma}{\alpha}, \quad \beta_1 = \sqrt{\beta^2 - \frac{\gamma^2}{\alpha^2}}. \quad (4.27)$$

From (4.8) and in the limit (2.6) (where the rôle of h is played here by g_1) we have that

$$\alpha_3 = rv_3 = r\sqrt{(x_1 + \psi)^2 + x_3^2} \implies v_3 = \sqrt{(x_1 + \psi)^2 + x_3^2} \quad (4.28)$$

and

$$\beta_1 = \frac{x_3 \cos \psi}{\sqrt{(x_1 + \psi)^2 + x_3^2}}, \quad \beta_3 = \frac{(x_1 + \psi) \cos \psi}{\sqrt{(x_1 + \psi)^2 + x_3^2}}, \quad \beta_0 = \sin \psi. \quad (4.29)$$

Due to (4.26) the entries b and c in (4.10) are zero. Therefore, the only non-vanishing contribution comes from the contributes from the term with $k = 0$ and $m_2 = n_2$. In addition, the entries a and d are, to leading order, of the form $a = d^* \simeq 1 + irv_3$. Hence, in this limit and for large j , we have that

$$\lim_{j \rightarrow \infty} a^{j-m+m_2} d^{j+m-m_2} = e^{-2i\delta v_3 \cos \zeta}. \quad (4.30)$$

Also note that the coefficients $A_{m-m_2, m-m_2, 0}^{j-n} = 1$. Therefore we obtain the finite sum

$$\Psi_{j_2, n, \delta}(x_1, x_3, \psi) = \lim_{j \rightarrow \infty} \Psi_{j-n, j_2}^j = \sum_{m_2=-j_2}^{j_2} \Gamma_{j_2, m_2, n, \delta}(v_3) R_{m_2, m_2}^{j_2}(g_2), \quad (4.31)$$

where

$$\begin{aligned} \Gamma_{j_2, m_2, n, \delta}(v_3) &= j \int_0^\pi d\zeta \sin \zeta \left(d_{m_2, n}^{j_2}(\zeta) \right)^2 e^{-2i\delta v_3 \cos \zeta} \\ &= j N_{\alpha\beta}^{j_2} \int_{-1}^1 dx (1-x)^\alpha (1+x)^\beta \left[P_{j-\alpha/2-\beta/2}^{\alpha, \beta}(x) \right]^2 e^{-2i\delta v_3 x}, \end{aligned} \quad (4.32)$$

with the definitions

$$\alpha = |m_2 - n|, \quad \beta = |m_2 + n|, \quad N_{\alpha\beta}^{j_2} = \frac{\left(j_2 + \frac{\alpha+\beta}{2}\right)! \left(j_2 - \frac{\alpha+\beta}{2}\right)!}{\left(j_2 + \frac{\alpha-\beta}{2}\right)! \left(j_2 + \frac{\beta-\alpha}{2}\right)!}. \quad (4.33)$$

The overall constant j will be subsequently dropped.

We present below some explicit examples:

For $j_2 = 0$:

$$\Psi_{0,0,\delta} = \frac{\sin 2\delta v_3}{\delta v_3} . \quad (4.34)$$

For $j_2 = \frac{1}{2}$:

$$\Psi_{1/2,\pm 1/2,\delta} = \pm \frac{\beta_3}{\delta v_3} \cos 2\delta v_3 + \frac{2\delta\beta_0 v_3 \mp \beta_3}{2\delta^2 v_3^2} \sin 2\delta v_3 . \quad (4.35)$$

For $j_2 = 1$:

$$\begin{aligned} \Psi_{1,\pm 1,\delta} &= \frac{\beta_1^2 - 2\beta_3(\beta_3 \mp 2\delta\beta_0 v_3)}{2\delta^2 v_3^2} \cos 2\delta v_3 \\ &\quad + \frac{2\beta_3^2 - \beta_1^2 + \mp 4\delta\beta_0\beta_3 v_3 + 4\delta^2(\beta_0^2 - \beta_3^2)v_3^2}{4\delta^2 v_3^3} \sin 2\delta v_3 , \\ \Psi_{1,0,\delta} &= \frac{2\beta_3^2 - \beta_1^2}{\delta^2 v_3^2} \cos 2\delta v_3 + \frac{\beta_1^2 - 2\beta_3^2 + 2\delta^2(1 - 2\beta_1^2)v_3^2}{2\delta^2 v_3^3} \sin 2\delta v_3 . \end{aligned} \quad (4.36)$$

We have checked that these are eigenfunctions of (3.1) and the corresponding eigenenergies agree with (4.21). Obviously the above expressions rapidly become quite complicated as the spin increases and it would have been difficult to compute, to say the least, without using the correspondence (2.7). In the limit $\delta \rightarrow 0$ the general state is given by the character of the representation of spin j_2 , as it was shown in [6], given by

$$\begin{aligned} \Psi_{j_2} &= \sum_{m=0}^{[j_2]} \frac{(2j_2 + 1)!}{(2m + 1)! (2j_2 - 2m)!} \beta_0^{2(j_2 - m)} (\beta_0^2 - 1)^m \\ &= 2^{2j_2} \beta_0^{2j_2} - 2^{2j_2 - 2} (2j_2 - 1) \beta_0^{2j_2 - 2} + \dots = U_{2j_2}(\beta_0) , \\ E_{j_2} &= \frac{j_2(j_2 + 1)}{k_2} , \end{aligned} \quad (4.37)$$

where $U_{2j_2}(\beta_0)$ is the Chebyshev polynomials of the 2nd kind. We have verified that the above eigenfunctions indeed solve the scalar equation (3.1) with the indicated eigenvalues.

5 Non-abelian duality in non-isotropic cases

The idea of using a limiting procedure to take advantage of symmetries in order to solve field equations can be rather straightforwardly extended to other σ -models in which there is a group theoretical structure.

In the rest of this paper we focus our attention on backgrounds in which the isometry group acts with no isotropy. In particular, consider the Principal Chiral Model (PCM) [25] for a group G . The σ -model action is given by

$$S(g) = -\frac{k}{\pi} \int_M \text{Tr}(g^{-1} \partial_- g g^{-1} \partial_+ g) . \quad (5.1)$$

This is invariant under the global $G_L \times G_R$ symmetry

$$g \rightarrow \Lambda_L^{-1} g \Lambda_R , \quad (\Lambda_L, \Lambda_R) \in G . \quad (5.2)$$

We would like to find the non-abelian dual of this action corresponding to a subgroup $H_L \in G_L$. We introduce gauge fields A_\pm in the corresponding Lie algebra and add the appropriate Lagrange multiplier. The corresponding action is

$$S_{\text{nonab}}(g, v, A_\pm) = -\frac{k}{\pi} \int_M \text{Tr}(g^{-1} D_- g g^{-1} D_+ g) + i \text{Tr}(v F_{+-}) , \quad (5.3)$$

with the covariant derivatives, corresponding to minimal coupling to the gauge fields, given by

$$D_\pm g = \partial_\pm g - A_\pm g . \quad (5.4)$$

The action above is invariant under the local "left" symmetry

$$g \rightarrow \lambda^{-1} g , \quad v \rightarrow \lambda^{-1} v \lambda , \quad A_\pm \rightarrow \lambda^{-1} A_\pm \lambda - \lambda^{-1} \partial_\pm \lambda , \quad \lambda(\sigma^+, \sigma^-) \in H , \quad (5.5)$$

as well as the global "right" (mainly) symmetry $g \rightarrow \lambda'^{-1} g \Lambda_R$, where $\Lambda_R \in G$ and λ' belongs to the maximal subgroup of G that commutes with the gauge group H .

Similarly to the discussion in section 2 we may reproduce (5.3) via a limiting procedure. We introduce an independent gauged WZW action for a group H and start with

$$S_{\text{Hyb}}(g, h, A_\pm) = -\frac{k}{\pi} \int_M \text{Tr}(g^{-1} D_- g g^{-1} D_+ g)$$

$$+ \ell I_0(h) + \frac{\ell}{\pi} \int_M \text{Tr} \left[A_- \partial_+ h h^{-1} - A_+ h^{-1} \partial_- h + A_- h A_+ h^{-1} - A_- A_+ \right]. \quad (5.6)$$

Then, the limit (2.6) reproduces the Lagrange multiplier term yielding (5.3).

The gauge fields in (5.3) are non-dynamical and, as before, they can be eliminated via their equations of motion. We also gauge fix $\dim(H)$ of the parameters. In this paper we are mainly interested in the case with $H = G$. Then we can choose the gauge $g = I$, completely getting rid of the parameters in G and being left with a σ -model solely for the Lagrange multipliers v . The result is

$$S = \frac{k}{\pi} \int \partial_+ v_a (K^{-1})^{ab} \partial_- v_b, \quad K_{ab} = \delta_{ab} + f_{ab}, \quad f_{ab} \equiv f_{ab}^c v_c. \quad (5.7)$$

The process of integrating out the gauge fields introduces an extra factor in the path integral measure given by

$$e^{-2\Phi} = \det(K), \quad (5.8)$$

which would have been the dilaton factor if there were a stringy origin of the background. Even though there is no such interpretation here, it is crucial that Φ be included in the measure of the scalar wave equation (3.1).

5.1 Asymmetric coset reduction

To obtain the states for the scalar sector of the above theory we follow a group theoretic procedure similar to the one used for the non-abelian duals of WZW in section 3.

Our starting point will be a general model of the type $G_{k_1}^{(1)} \times G_{k_2}^{(2)} / H_{k_1+k_2}$, where the base manifold G is the product of two groups $G^{(1)}$ and $G^{(2)}$ and the gauged group H is a subgroup of both $G^{(1)}$ and $G^{(2)}$. The group reduction is done in an asymmetric way. Specifically, the configuration space is parametrized by the two group elements (g_1, g_2) modulo the identification

$$(g_1, g_2) \sim (h g_1, h g_2 h^{-1}), \quad g_i \in G^{(i)}, \quad i = 1, 2, \quad h \in H. \quad (5.9)$$

Clearly for $G^{(2)} = H$ this reproduces the structure of (5.6) of the previous section.

We start again with the set of eigenstates of the Laplacian on the base group manifold, which are given by the matrix elements of direct products of two irreps of the $G^{(i)}$'s,

$R^{(1)} \times R^{(2)}$:

$$R_{\alpha\beta}^{(1)}(g_1) R_{\mu\nu}^{(2)}(g_2) , \quad (5.10)$$

with eigenvalues

$$E(R) = \frac{C_2^{(1)}(R)}{k_1} + \frac{C_2^{(2)}(R)}{k_2} . \quad (5.11)$$

Under the H -transformation defining the coset manifold, the above states transform in the representation

$$R^{(1)} \times R^{(2)} \times \bar{R}^{(2)} , \quad (5.12)$$

with the indices α , μ and ν transforming under the respective group factor, index β remaining free. The above direct product must be projected to the singlets of H . Decomposing $R^{(1)}$ into irreps r_{1i} and $R^{(2)}$ into irreps r_{2j} of H , we must reduce irreps of the form

$$r_{1i} \times r_{2j} \times \bar{r}_{2k} , \quad (5.13)$$

into singlets. This will be possible, in general, only for specific choices of i, j, k ; specifically, whenever the decomposition of $r_{1i} \times r_{2j}$ contains r_{2k} . Assuming this to be the case, we denote $C_{\alpha\mu}^a(R^{(1)}, R^{(2)}; r_{2j})$ one of the Clebsch–Gordan coefficients projecting the state α of $R^{(1)}$ and the state μ of $R^{(2)}$ into the state a of r_i (there could be many, as the product $R^{(1)} \times R^{(2)}$ may contain more than one copies of r_{2i}). We also denote by $P_v^a(R^{(2)}, r_{2j})$ the projector that projects the state ν of $R^{(2)}$ onto the state a of r_{2j} . We can then construct coset eigenstates as

$$\psi_{R^{(1)}, R^{(2)}, r|\beta}(g_1, g_2) = \sum_{a;\alpha,\mu,\nu} C_{\alpha\mu}^a(R^{(1)}, R^{(2)}; r) P_v^a(R^{(2)}, r) R_{\alpha\beta}^{(1)}(g_1) R_{\mu\nu}^{(2)}(g_2) . \quad (5.14)$$

The above states have two degeneracy indices: β , which, as we stated, does not participate in the H -transformation and remains free, and an (unmarked) extra index enumerating the various copies of r contained in $R^{(1)} \times R^{(2)}$.

A special case of interest, analogous to the vector coset case studied in the previous paper, is the one where $G^{(1)} = G^{(2)} = H \equiv G$. Then the decomposition of $R^{(1)}$ and $R^{(2)}$ into irreps r_i is not required and we only need the Clebsch–Gordan coefficients $C_{\alpha\mu}^\nu(R^{(1)}, R^{(2)})$ projecting the state α of $R^{(1)}$ and the state μ of $R^{(2)}$ into the state ν of $R^{(2)}$ (assuming $R^{(1)} \times R^{(2)}$ contains $R^{(2)}$). These are the same as the Clebsch–Gordan coefficients $C_{\mu\nu}^\alpha(R^{(1)}, R^{(2)})$ projecting the state μ of $R^{(2)}$ and the state ν of $\bar{R}^{(2)}$ into the

state α of $R^{(1)}$. The eigenstates are

$$\psi_{R^{(1)}, R^{(2)}; \beta}(g_1, g_2) = \sum_{\alpha, \mu, \nu} C_{\alpha\mu}^\nu(R^{(1)}, R^{(2)}) R_{\alpha\beta}^{(1)}(g_1) R_{\mu\nu}^{(2)}(g_2). \quad (5.15)$$

Further, since $G^{(1)} = H$, we can use the gauge symmetry to completely gauge fix the first field to $g_1 = \mathbb{I}$ (as was done previously to arrive at (5.7)) , in which case $R_{\alpha\beta}^{(1)}(g_1 = \mathbb{I}) = \delta_{\alpha\beta}$. Then, the expression for the eigenstates simplifies further to

$$\psi_{R^{(1)}, R^{(2)}; \beta}|_{\text{g.f.}} = \sum_{\mu, \nu} C_{\beta\mu}^\nu(R^{(1)}, R^{(2)}) R_{\mu\nu}^{(2)}(g_2). \quad (5.16)$$

For $R^{(1)} = \mathbb{I}$ (singlet) the summation becomes a trace and the states become the (conjugation-invariant) characters of G .

The identification of irreps $R^{(1)}$ and $R^{(2)}$ such that their product contain $R^{(2)}$ is a matter of group theory for the group G . For $G = SU(N)$ an obvious constraint is that $R^{(1)}$ should have zero Z_N charge (the number of boxes in its Young tableau should be a multiple of N).

As an aside, we note that exactly the same group theory selection rules and eigenstates arise in the case of the spin-Calogero–Sutherland model obtaining from matrix models or two-dimensional Yang–Mills theory, in which case $R^{(1)}$ is the irrep of $SU(N)$ that encodes the spins of the particles while $R^{(2)}$ is the irrep that generates the various energy eigenstates [26].

6 Example: Non-abelian dual of the $SU(2)$ PCM

In this case we have representation matrices and group structure constants given by

$$t_a = \frac{\sigma_a}{\sqrt{2}}, \quad f_{abs} = \sqrt{2}\epsilon_{abc}, \quad (6.1)$$

where σ_a are the standard Pauli-matrices. Rescaling as $v_a \rightarrow v_a/\sqrt{2}$ we have that

$$K_{ab} = \delta_{ab} + \epsilon_{abc}v_c \implies (K^{-1})^{ab} = \frac{1}{1+v^2}(\delta_{ab} + v_av_b - \epsilon_{abc}v_c). \quad (6.2)$$

Thus we obtain a σ -model (essentially the one computed in [9, 13]) with metric

$$ds^2 = (\delta_{ab} + v_a v_b) \frac{dv_a dv_b}{1 + v^2} \quad (6.3)$$

and antisymmetric tensor

$$B_{ab} = -\frac{\epsilon_{abc} v_c}{1 + v^2} . \quad (6.4)$$

The dilaton factor is simply $e^{-2\Phi} = 1 + v^2$. This σ -model has by construction an $SU(2)$ symmetry corresponding to rotations of the coordinates v_a that can be made manifest by introducing spherical coordinates in place of the Cartesian ones. In particular, for the metric we obtain (for notational conformity v is replaced by r)

$$ds^2 = dr^2 + \frac{r^2}{1 + r^2} d\Omega_2^2 . \quad (6.5)$$

This is a smooth space, due to the fact that the isometry acts with no isotropy, interpolating between \mathbb{R}^3 and $\mathbb{R} \times S^2$.

6.1 Solving the wave equation

The scalar wave equation (3.1) for the above background can be solved explicitly. Indeed, using spherical coordinates and writing $\Psi = \psi(r) Y_{l,m}(\theta, \phi)$, where the $Y_{l,m}$'s are the standard spherical harmonics, we find that the radial function obeys

$$\frac{d^2 \psi}{dr^2} + \frac{2}{r} \frac{d\psi}{dr} + \left(k^2 - \frac{l(l+1)}{r^2} \right) \psi = 0 , \quad (6.6)$$

where

$$E = k^2 + l(l+1) , \quad k \in \mathbb{R} . \quad (6.7)$$

This is the spherical Bessel equation with solutions regular at the origin the corresponding functions $j_l(kr)$. Hence, the full solution is given by

$$\Psi_{l,m,k}(r, \theta, \phi) = j_l(kr) Y_{l,m}(\theta, \phi) , \quad (6.8)$$

which constitute a complete set of orthogonal eigenfunctions.¹

We would like to use (5.16) in the appropriate limit that we have described in order to

¹For non-abelian T-duals corresponding to PCM for higher than $SU(2)$ groups the eigenfunctions are not necessarily completely separable.

recover solutions of the wave equation corresponding to the non-abelian model and in particular the one given above.

Putting $R^{(1)} = (\text{spin } j_1)$ and $R^{(2)} = (\text{spin } j)$, and using the gauge $g_1 = \mathbb{I}$ as in (5.16), we have

$$\Psi_{j,j_1,n_1}|_{\text{g.f.}} = \sum_m C_{j_1,n_1,j,m-n_1}^{j,m} R_{m-n_1,m}^j(g_2), \quad (6.9)$$

The requirement $R^{(2)} \in R^{(1)} \times R^{(2)}$, which in the present $SU(2)$ case means $j \in j_1 \times j$, imposes the condition $|j - j_1| \leq j \leq j + j_1$, or in other words $0 \leq j_1 \leq 2j$. Since we are eventually interested in the limit of large j but finite j_1 , this is not a restriction. In addition, j_1 must necessarily be an integer. The spin index n_1 in the above, taking $2j_1 + 1$ values, plays the role of the free (degeneracy) index β of (5.16).

Then for g_2 we have to use the limit (2.6) and at the same time take the large spin limit. Taking the limit (2.6) in the parametrization (4.3) of the group element g_2 , implies that²

$$\phi_1 \simeq \frac{1}{2\ell} v_3, \quad \theta \simeq \frac{1}{\ell} \sqrt{v_1^2 + v_2^2}, \quad \phi_2 = -\tan^{-1} \frac{v_1}{v_2}. \quad (6.10)$$

As previously we will need a linear relation similar to (4.20)

$$\ell = \frac{j}{\delta}, \quad \delta \in \mathbb{R}^+, \quad (6.11)$$

so that the infinite level is directly related to the infinite spin limit. In addition, let

$$m = js = j \cos \zeta, \quad |s| \leq 1, \quad 0 \leq \zeta \leq \pi. \quad (6.12)$$

We compute next the $j \rightarrow \infty$ limit of $R_{m-n_1,m}^j$ using the expression (4.11) for it. First we see that the overall exponential factor becomes

$$\lim_{j \rightarrow \infty} e^{-i(m-m_1)(\phi_1-\phi_2)+m(\phi_1+\phi_2)} = e^{-i\phi_2 n_1} e^{-i\delta s v_3}. \quad (6.13)$$

Taking the corresponding limit in $d_{m-n_1,m}^j$, as given by (4.13), requires extra care. First, we note that using Stirling's formula in the form

$$(n+a)! \simeq \sqrt{2\pi n} n^{n+a} e^{-n}, \quad \text{for } n \gg 1, \quad a = \text{finite}, \quad (6.14)$$

²To avoid proliferation of symbols we set the overall constant on the PCM action (5.1) to one. This way it is also not confused with k introduced in (6.6).

we have that

$$\lim_{j \rightarrow \infty} \frac{(j + m/2 + n/2)! (j - m/2 - n/2)!}{(j - m/2 + n/2)! (j + m/2 - n/2)!} = \left(\frac{1 + |s|}{1 - |s|} \right)^{|n_1|}, \quad (6.15)$$

where, using the definitions (4.12), we have made in the left hand side the replacements $m \rightarrow |n_1|$ and $n \rightarrow |2m - n_1| \simeq 2|m|$. The limiting behaviour of the Jacobi polynomials, also appearing in (4.13), is found in terms of Bessel functions. In general one may show that

$$\lim_{j \rightarrow \infty} P_{\delta j}^{\alpha, \gamma j + \beta - \alpha} \left(\cos \frac{x}{j} \right) = \left(\frac{2\delta j}{\sqrt{\delta(\delta + \gamma)x}} \right)^\alpha J_\alpha \left(\sqrt{\delta(\delta + \gamma)x} \right), \quad (6.16)$$

where α, β, γ and δ (not to confuse it with δ introduced in (6.11)) are finite constants.³ This identity can be proven by first taking the limit in the differential equation obeyed by the Jacobi polynomials and showing that after changing variables, as indicated by the above expression, it reduces to the Bessel equation. The overall normalization constant can be fixed by examining the behaviour of both sides of it at small values of x .⁴ Assembling everything we find from (4.13) that

$$\lim_{j \rightarrow \infty} d_{m-n_1, m}^j = J_{|n_1|} \left(\delta \sqrt{1 - s^2} \sqrt{v_1^2 + v_2^2} \right). \quad (6.17)$$

In addition, using (4.24) and the property of the Clebsch–Gordan coefficients under interchanging the order in the spin pairs in the lower row, we obtain that

$$\lim_{j \rightarrow \infty} C_{j_1, n_1, j, m - n_1}^{j, m} = (-1)^{j_1} d_{n_1, 0}^{j_1}(\zeta) \sim (1 - s^2)^{|n_1|/2} P_{j_1 - |n_1|}^{|n_1|, |n_1|}(s) \sim P_{j_1}^{n_1}(s), \quad (6.18)$$

where we have used (4.13) and, in the last step, the well known relation between the Jacobi polynomials and the associated Legendre functions. Altogether, omitting a constant overall factor, we obtain

$$\Psi_{j_1, n_1}(r, \theta, \phi) = e^{-in_1 \phi_2} \int_{-1}^1 ds e^{-i\delta v_3 s} J_{|n_1|} \left(\delta \sqrt{1 - s^2} \sqrt{v_1^2 + v_2^2} \right) P_{j_1}^{n_1}(s). \quad (6.19)$$

This should be a solution of the scalar equation (3.1) and in particular it should be

³In our case $\alpha = |n_1|, \gamma = 2|s|, \delta = 1 - |s|$ and $\beta - \alpha = \mp n_1$, where the upper (lower) sign agrees with the sign of m (equivalently s).

⁴In a slight generalization of (6.16) one replaces the argument of the Jacobi polynomial $\cos \frac{x}{j}$ with any function behaving as $1 - x^2/(2j^2)$ for large j .

just (6.8). The comparison should be made using the spherical coordinates (r, θ, ϕ) , in place of the v_i 's. After also changing integration variable we obtain

$$\Psi_{j_1, n_1}(r, \theta, \phi) = e^{-in_1\phi} \int_0^\pi d\zeta \sin \zeta e^{-i\delta r \cos \theta \cos \zeta} J_{|n_1|}(\delta r \sin \theta \sin \zeta) P_{j_1}^{n_1}(\cos \zeta). \quad (6.20)$$

The last integral has been computed in [27] and is proportional to $j_{j_1}(\delta r) P_{j_1}^{n_1}(\cos \theta)$. Taking into account the overall exponential factor we arrive at (6.8), with the obvious identification of the quantum numbers. In particular, we see that the (integer) spin j_1 plays the role of the angular momentum l in (6.8).

This completes the proof of the recovery of the solutions of the model from the large-spin limit of the appropriate gauged model.

7 Concluding remarks and future directions

In this paper we argued that an appropriate high-level/high-highest weight limit in parent gauged σ -models reproduces the spectrum of appropriate non-abelian T-duals. We demonstrated this by explicitly working out examples where the involved group structures are based on $SU(2)$. We focused on non-abelian T-duals of WZW models and PCM, but we believe that our findings can be extended to all σ -models with non-abelian isometries and their T-duals. However, we do not have a general proof of that statement.

It will be interesting to explore further the limiting procedure that we have established. In particular, for the case of non-abelian duals of WZW models it should be possible to carry it out in full detail at the exact conformal field theory level.

We conclude by pointing out that the connection to matrix models and integrable systems of the Calogero type, noticed already in the previous paper [6], persists in the cases studied presently. Specifically, the states and spectrum identified in section 5.1 for the case $G^{(1)} = G^{(2)} = H$ are the same as those of spin-Sutherland models. The possibility for further relations between WZW models and generalized integrable models remains open and worth exploring.

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